1 Basic Rules

• Multiplication
(1) $a \cdot b = b \cdot a$ (commutativity)
(2) $1 \cdot a = a = a \cdot 1$ (identity element)
(3) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)
(4) $a \cdot \frac{1}{a} = 1$ if $a \neq 0$ (inverse element)
Note 1: Rule (4) shows that "division" is an inverse operation of multiplication.

• Addition
(1) $a + b = b + a$ (commutativity)
(2) $0 + a = a = a + 0$ (identity element)
(3) $(a + b) + c = a + (b + c)$ (associativity)
(4) $a + (-a) = 0$ (inverse element)
Note 2: Rule (4) says that "subtraction" is the inverse operation of addition.
Note 3: Therefore, if one accepts numbers of the form $(\frac{1}{a})$ if $a \neq 0$, and $(-a)$, one needs to pay attention only to the two operations: multiplication and addition (instead of four operations: addition, subtraction, multiplication and division).

• Sign Law:
$-(+a) = +(-a) = -a$
$+(+a) = a = -(-a)$
$a \cdot (-b) = -a \cdot b$
$(-a) \cdot (-b) = a \cdot b$

• Distribution Law
$a \cdot (b + c) = a \cdot b + a \cdot c$
Hence we also have the following:
$(b + c) \cdot a = b \cdot a + c \cdot a$ and
$a \cdot (b - c) = a \cdot (b + (-c)) = a \cdot b + a \cdot (-c) = a \cdot b - a \cdot c$

• Basic operations involving fractions:
(1) $\frac{a}{b} = a \cdot \frac{1}{b}$, if $b \neq 0$ (note: $\frac{a}{0}$ is not defined.)
(2) $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$
(3) $\frac{a}{a} = 1$, if $a \neq 0$ (this is law (4) of multiplication.)
(4) $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$ (dividing by $\frac{c}{d}$ means multiplication by $\frac{d}{c}$.)

For example, $\frac{a}{b} \div \frac{a}{b} = \frac{a}{b} \cdot \frac{b}{a} = \frac{a \cdot b}{b \cdot a} = \frac{a \cdot b}{a \cdot b} = 1$, which shows also that

$$\frac{\frac{a}{b}}{\frac{a}{b}}$$

should be 1.

• Important convention: In an arithmetic expression, always simplify parentheses and "$\cdot, \div$" first. Then do "$+, -$".
For example, $(-2 + 3 \cdot 2) - 3 \cdot 5 = (-2 + 6) - 15 = 4 - 15 = -11$.

• Application
1. Review GCF and LCM (or LCD) from the text.
2. To reduce a fraction $\frac{a}{b}$ to the lowest terms, you can either
(1) divide a and b by their GCF, or
(2) divide a and b by any common factor of a and b, until the resulting numerator and the denominator have no common factors, or
(3) factorize a and b in terms of prime factors, then cancel the common terms.

For example,
(1) $\frac{24}{36} = \frac{24/12}{36/12} = \frac{2}{3}$ ($\because$ GCF of 24 and 36 is 12.)
(2) $\frac{24}{36} = \frac{24/2}{36/2} = \frac{12}{18} = \frac{12/2}{18/2} = \frac{6}{9} = \frac{6/3}{9/3} = \frac{2}{3}$.
(3) $\frac{24}{36} = \frac{2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 3} = \frac{2}{3}$.

3. To add or subtract fractions, one needs to find their common denominators first.
For example, $\frac{2}{3} + \frac{3}{5} = \frac{2 \cdot 5}{3 \cdot 5} + \frac{3 \cdot 3}{3 \cdot 5} = \frac{10}{15} + \frac{9}{15} = \frac{19}{15}$.
Note 4: $\frac{a}{b} = \frac{a}{b} \cdot \frac{c}{c}$ (since $\frac{c}{c} = 1$ and multiplication by 1 doesn't change the number.)

For example, $\frac{2}{3} = \frac{2}{3} \cdot \frac{5}{5} = \frac{10}{15}$, and $\frac{3}{5} = \frac{3}{5} \cdot \frac{3}{3} = \frac{9}{15}$.

2 Solving a linear equation of one variable

The idea of solving linear equation of one variable is pretty simple: We want to reduce it to the following form first:

$$a \cdot x = b,$$

where x is the variable we want to solve, and a and b are known numbers. (Namely, we collect the terms involving x on the left side of the equality and the terms not involving x on the right side of the equality.)

Then we divide into the following cases:
(1) If a and b are both zero, namely if you get $0 = 0$, then the equation is always true, no matter what value you assign for x. Therefore in this case, x can be any number and the equation has infinitely many solutions.
(2) If $a = 0$ and $b \neq 0$, then you get $0 = b \neq 0$, which is absurd. Therefore in this case, the equation has no solutions.
(3) If $a \neq 0$, you can divide on both sides of the equation by a (or what amounts to the same thing, multiply both sides by $\frac{1}{a}$) to get

$$\frac{a \cdot x}{a} = a \cdot \frac{1}{a} \cdot x = 1 \cdot x = x = \frac{b}{a},$$

which shows that $x = \frac{b}{a}$ is the solution of the equation.

For example,
(1) Solve $2x + 3 = 0$ for x.
solution: Add -3 to both sides of the equality to get $2x+3+(-3) = 0+(-3)$, hence get $2x = -3$. Then divide both sides of the equality by 2 to get $x = \frac{-3}{2} = -\frac{3}{2}$. DONE.

(2) Solve $(2x - 3 \cdot 2) + 3(x + 5) = 4x - 5$ for x.
solution: Simplify the above expression using the basic rules introduced in the first section, for example apply the distribution law if necessary. For example, the following procedure would do.
The above equality is equivalent to $2x - 6 + 3x + 15 = 4x - 5$, which gives $5x + 9 = 4x - 5$. Adding $-4x$ to both sides then gives $x + 9 = -5$, which after subtraction by 9 from both sides gives $x = -14$. DONE.

(3) Solve $3 \cdot (x - 2) = 5$ for x.
We can do this in at least two different ways.
1st method: Divide both sides of the equality by 3 to get $(x - 2) = \frac{5}{3}$. Then adding 2 to both sides gives $x - 2 + 2 = \frac{5}{3} + 2 = \frac{5}{3} + \frac{6}{3} = \frac{11}{3}$, and so $x = \frac{11}{3}$. DONE.
2nd method: Expand the left side by distribution law to get $3x - 6 = 5$. Then add 6 to both sides to get $3x = 11$. Then divide both sides as usual to get $x = \frac{11}{3}$. DONE.

3 Practice Problems

Make sure you can do all of the following problems. Please let me know if either you don't know how to do them or if your answers are different from mine. (See next section for answers.)

1. Factorize the following numbers into prime factors:
(1) 36
(2) 45
(3) 39
(4) 48
(5) 250

2. Find the GCF of the numbers $36, 45$, and 48. Find the LCM of them.

3. Simplify the following fractions to the lowest terms:
(1) $\frac{24}{35}$

(2) $\frac{24}{36}$

(3) $\frac{18}{30}$

(4) $\frac{84}{1470}$

(5) $\frac{15^2}{3^2 \cdot 5^3}$

4. Find the least common denominator (LCD) of the following fractions: $\frac{7}{6}$, $\frac{2}{15}$ and $\frac{3}{35}$. Then find the sum of these three fractions and write your answer in lowest terms.

5. Solve $3x - 8 = 1$ for x.
6. Solve $5x = 18$ for x.
7. Solve $-2x = 38$ for x.
8. Solve $2x + 3 = 4x - 8 - 2x$ for x.
9. Solve $x - 8 + 6x = -4 + 12 - x - 16 + 8x$ for x.
10. Solve $2(x + 5) - 3 = 8$ for x.
11. Solve $3(x - 3) + 6x = -3(x - 2) + 5x - 6$ for x.
12. Solve $\frac{1}{3}(x - 5) + 3x = \frac{3}{2} + \frac{1}{2}x$ for x.

4 Answers to Practice Problems

1.
(1) $36 = 2 \cdot 2 \cdot 3 \cdot 3$.

(2) $45 = 3 \cdot 3 \cdot 5$.
(3) $39 = 3 \cdot 13$.
(4) $48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$.
(5) $250 = 2 \cdot 5 \cdot 5 \cdot 5 \cdot 5$.
2.
The GCF of $36, 45$ and 48 is 3.
The LCM of $36, 45$ and 48 is 720.
3.
(1) $\frac{24}{35} = \frac{24}{35}$

(2) $\frac{24}{36} = \frac{2}{3}$.

(3) $\frac{18}{30} = \frac{3}{5}$.

(4) $\frac{84}{1470} = \frac{2}{35}$

(5) $\frac{15^2}{3^2 \cdot 5^3} = \frac{1}{5}$.

4. The LCD of the three fractions is 210. The sum in lowest terms is $\frac{97}{70}$.

5. $x = 3$.

6. $x = \frac{18}{5}$.

7. $x = -19$.

8. No solutions.

9. x can be any number. (There are infinitely many solutions.)

10. $x = \frac{1}{2}$.

11. $x = \frac{9}{7}$.

12. $x = \frac{19}{17}$.